Efficient recognition of totally nonnegative cells

Seminar of Representation Theory and Related Areas: Third Workshop

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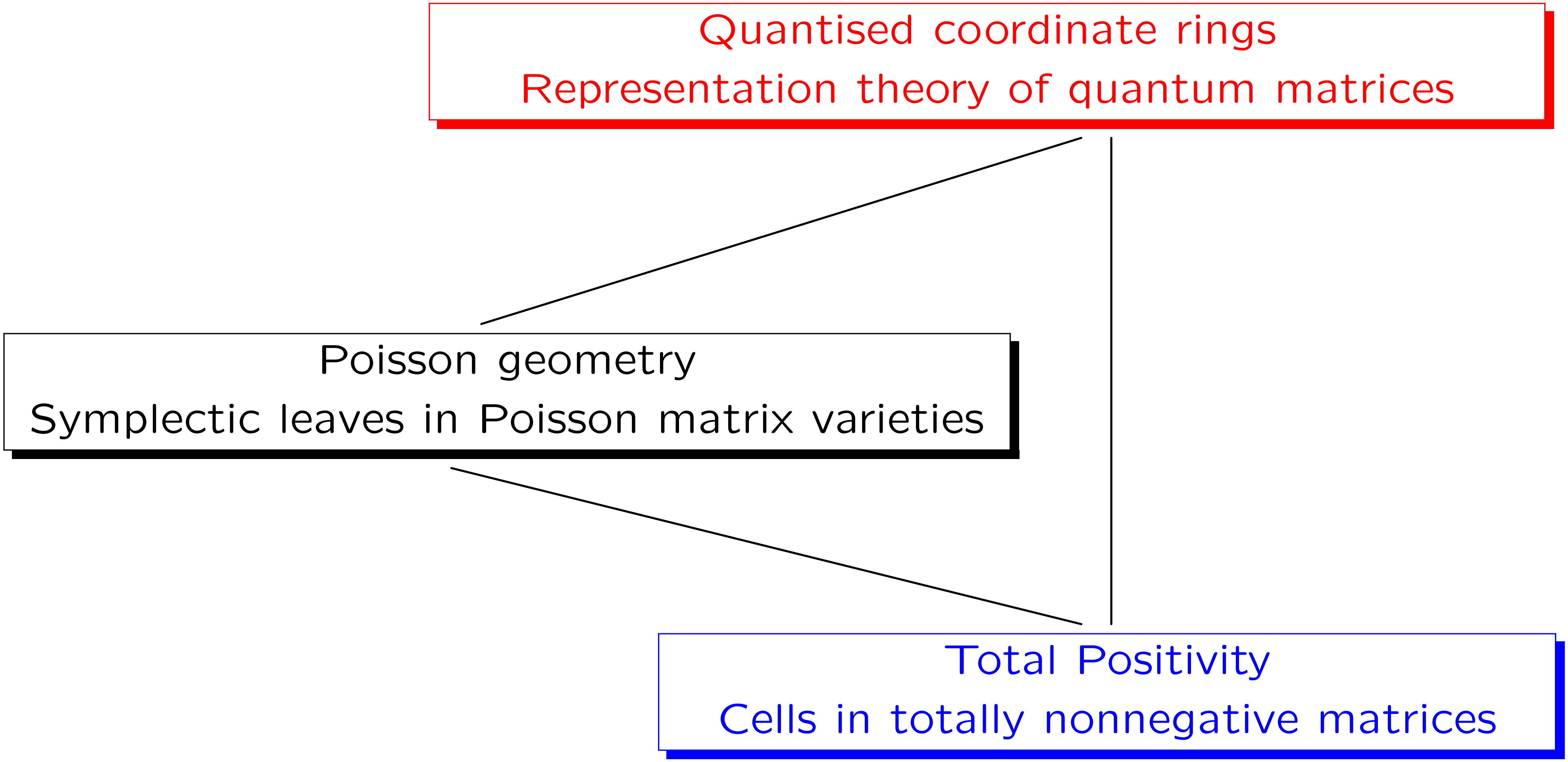
http://www.kent.ac.uk/ims/personal/sl261/index.htm


Quantised coordinate rings
Representation theory of quantum matrices
Poisson geometry
Symplectic leaves in Poisson matrix varieties
Total Positivity
Cells in totally nonnegative matrices

The nonnegative world

- A matrix is **totally positive** if each of its minors is positive.

- A matrix is **totally nonnegative** if each of its minors is nonnegative.

History

- Fekete (1910s)
- Gantmacher and Krein, Schoenberg (1930s): small oscillations, eigenvalues
- Karlin and McGregor (1950s): statistics, birth and death processes
- Lindström (1970s): planar networks
- Gessel and Viennot (1985): binomial determinants, Young tableaux
- Gasca and Peña (1992): optimal checking
- Lusztig (1990s): reductive groups, canonical bases
- Fomin and Zelevinsky (1999/2000): survey articles (eg Math Intelligencer)
- Postnikov (2007): the totally nonnegative grassmannian

Examples

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

¿ How much work is involved in checking if a matrix is totally positive?

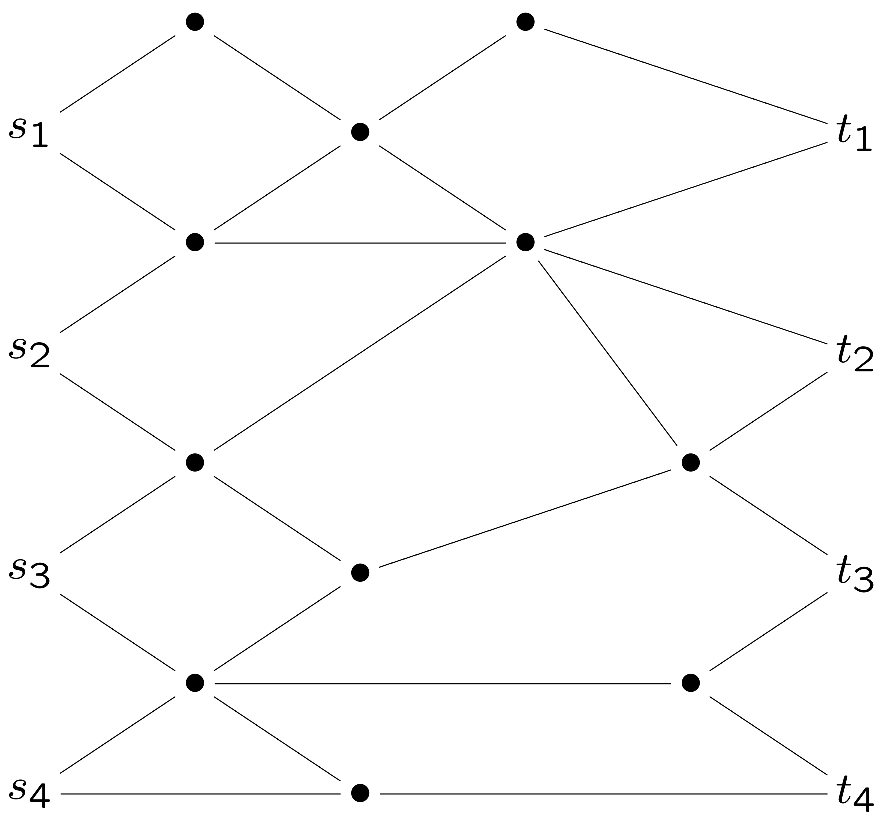
Eg. $n = 4$: we need to compute 69 minors.

$$\#\text{minors} = \sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n} - 1 \approx \frac{4^n}{\sqrt{\pi n}}$$

by using Stirling's approximation

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$

Planar networks Consider a directed graph with no directed cycles, n sources and n sinks.



Edges directed left to right.

$M = \left(m_{ij}\right)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Notation The minor formed by using rows from a set I and columns from a set J is denoted by $[I \mid J]$.

Theorem (Lindström)
The path matrix of any planar network is totally nonnegative. In fact, the minor $[I \mid J]$ is equal to the number of families of non-intersecting paths from sources indexed by I and sinks indexed by J.

If we allow weights on paths then even more is true.

Theorem (Brenti)
Every totally nonnegative matrix is the weighted path matrix of some planar network.

2×2 **case**

The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has **five** minors: $a, b, c, d, \Delta := ad - bc$.

If $b, c, d, \Delta = ad - bc > 0$ then

$$a = \frac{\Delta + bc}{d} > 0$$

so it is sufficient to check **four** minors.

Testing Total Positivity

Theorem (Fekete, 1913) A matrix is totally positive if each of its **solid minors** is positive.

Solid minors: $[i+1, ..., i+t \mid j+1, ..., j+t]$.

Examples: $[1, 2, 3 \mid 2, 3, 4]$ and $[2, 3, 4 \mid 2, 3, 4]$ are solid, whereas $[1, 2, 4 \mid 1, 2, 3]$ isn't.

Theorem (Gasca and Peña, 1992) A matrix is totally positive if each of its **initial minors** is positive.

Initial minors: solid minors with $i = 0$ or $j = 0$.

Examples: $[1, 2, 3 \mid 2, 3, 4]$ is initial, whereas $[2, 3, 4 \mid 2, 3, 4]$ isn't.

Question: What about TNN matrices?

Totally nonnegative cells

Let $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ be the set of totally nonnegative $m \times p$ real matrices.

Let Z be a subset of minors. The **cell** S_Z^o is the set of matrices in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ for which the minors in Z are zero (and those not in Z are nonzero).

Some cells may be empty. The space $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ is partitioned by the non-empty cells.

Example: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is TNN and belongs to the cell $S^{\circ}_{\{[12|12]\}}$.

A trivial example In $\mathcal{M}^{\text{tnn}}_{2,1}$, there are only 2 minors: [1|1] and [2|1]. Hence there are 2^2 cells:

$$S^{\circ}_{\{\emptyset\}} = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y > 0\}.$$

$$S^{\circ}_{\{[1|1]\}} = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \mid y > 0\}.$$

$$S^{\circ}_{\{[2|1]\}} = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x > 0\}.$$

$$S^{\circ}_{\{[1|1],[2|1]\}} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}.$$

Note that there are no empty cell.

Example In $\mathcal{M}_2^{\text{tnn}}$ the cell $S^\circ_{\{[2|2]\}}$ is empty.

For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is tnn and $d = 0$.

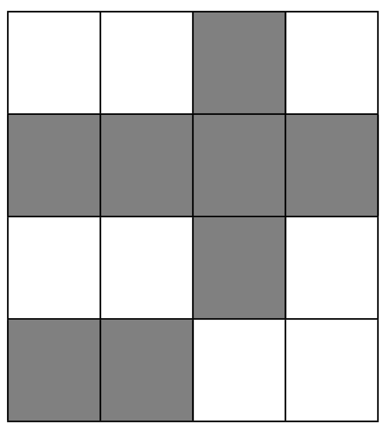
Then $a, b, c \geq 0$ and also $ad - bc \geq 0$.

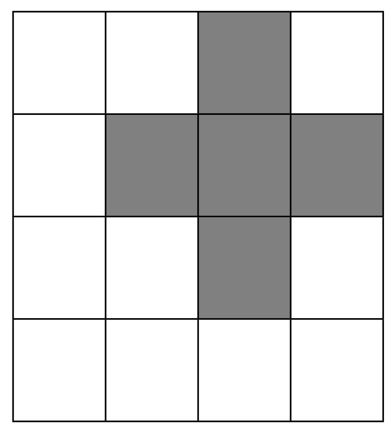
Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$.

Exercise There are 14 non-empty cells in $\mathcal{M}_2^{\text{tnn}}$.

Cauchon diagrams

A **Cauchon diagram** on an $m \times p$ array is an $m \times p$ array of squares coloured either black or white such that for any square that is coloured black the following holds:
Either each square strictly to its left is coloured black, or each square strictly above is coloured black.
Here are an example and a non-example

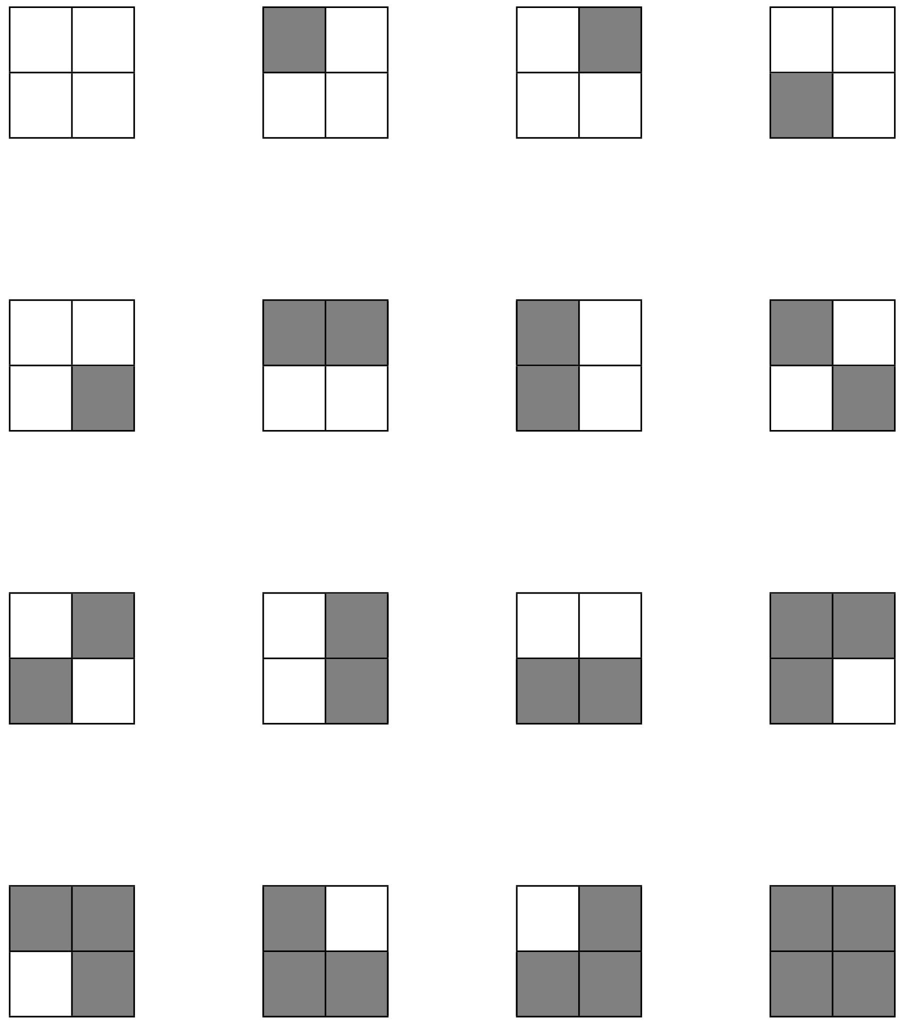




- **Postnikov (arXiv:math/0609764)** There is a bijection between Cauchon diagrams on an $m \times p$ array and non-empty cells S_Z° in $\mathcal{M}_{m,p}^{\text{tnn}}$.

For 2×2 **matrices**, this says that there is a bijection between Cauchon diagrams on 2×2 arrays and non-empty cells in $\mathcal{M}_2^{\text{tnn}}$.

2×2 **Cauchon Diagrams**



A first link between TNN and Cauchon diagrams

Let C be a Cauchon diagram. We say that $(i,\alpha) \in C$ if (i,α) is black in C

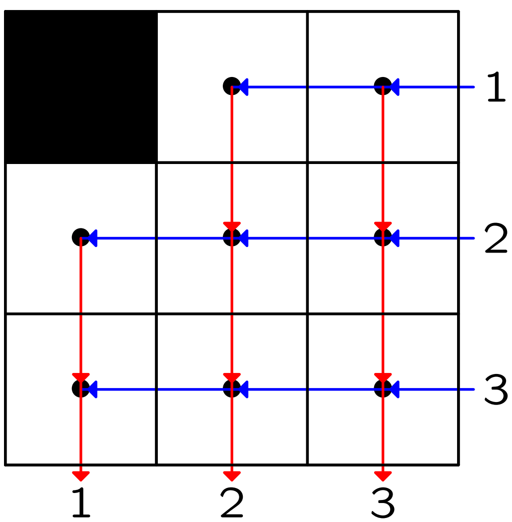
We say that $X = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(\mathbb{R})$ is a Cauchon matrix associated to the Cauchon diagram C provided that for all $(i,\alpha) \in [1,m] \times [1,p]$, we have $x_{i,\alpha} = 0$ if and only if $(i,\alpha) \in C$.

Lemma Every totally nonnegative matrix over $\mathbb{R}$ is a Cauchon matrix.

Proof Let $X = (x_{i,\alpha})$ be a tnn matrix. Suppose that some $x_{i,\alpha} = 0$, and that $x_{k,\alpha} > 0$ for some $k < i$. Let $\gamma < \alpha$. We need to prove that $x_{i,\gamma} = 0$. As X is tnn, we have $-x_{k,\alpha}x_{i,\gamma} = \det \begin{pmatrix} x_{k,\gamma} & x_{k,\alpha} \\ x_{i,\gamma} & x_{i,\alpha} \end{pmatrix} \geq 0$. As $x_{k,\alpha} > 0$, this forces $x_{i,\gamma} \leq 0$. But since X is tnn, we also have $x_{i,\gamma} \geq 0$, so that $x_{i,\gamma} = 0$, as desired.

Postnikov's Algorithm starts with a Cauchon diagram and produces a planar network. The family of minors associated to this Cauchon diagram is the set of minors that vanish on the path matrix associated to this planar network. The associated TNN cell is nonempty.

Example



$$\begin{pmatrix} 5 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

This path matrix is TNN by Lindström Lemma.
The only minor that vanishes is $[123|123]$.
So $\{[123|123]\}$ defines a nonempty cell.

Deleting Derivations Algorithm
= Cauchon reduction

Two algorithms

Deleting derivations algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a - bd^{-1}c & b \\ c & d \end{pmatrix}$$

Restoration algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a + bd^{-1}c & b \\ c & d \end{pmatrix}$$

Step (j, β)

Fix a row-index j and a column-index β. We define a map

$$f_{j,\beta} : \mathcal{M}_{m,p}(K) \to \mathcal{M}_{m,p}(K)$$

by

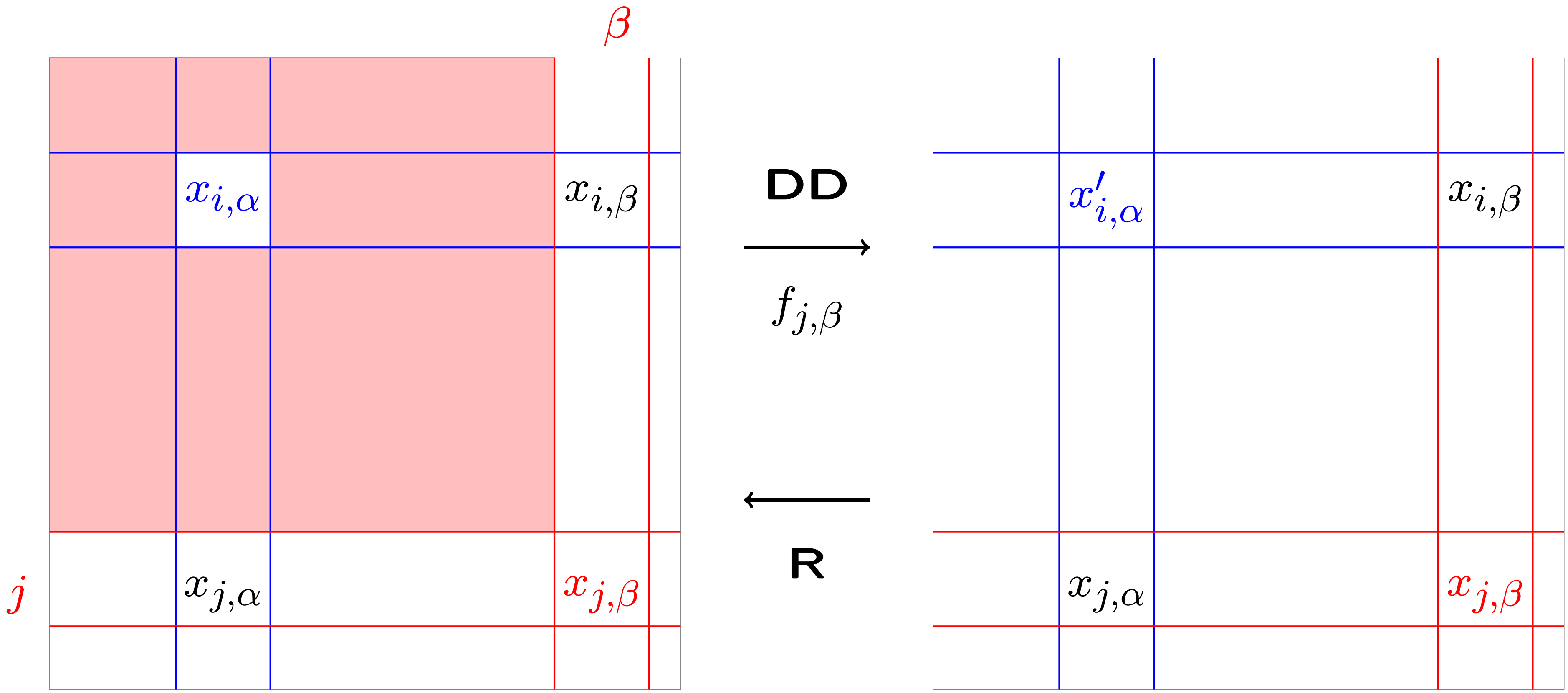
$$f_{j,\beta}((x_{i,\alpha})) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K),$$

where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} - x_{i,\beta} x_{j,\beta}^{-1} x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(k,\gamma)} := f_{k,\gamma} \circ \cdots \circ f_{m,p-1} \circ f_{m,p}(M)$.

$M^{(1,1)}$ is called the matrix obtained from M by the Deleting Derivations Algorithm.



with $x'_{i,\alpha} := x_{i,\alpha} - x_{i,\beta} x_{j,\beta}^{-1} x_{j,\alpha}$

ie $x_{i,\alpha} := x'_{i,\alpha} + x_{i,\beta} x_{j,\beta}^{-1} x_{j,\alpha}$

An example

Set $M = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then $M^{(3,3)} = f_{3,3}(M)$. The pivot is the entry in position $(3,3)$. The pivot is nonzero, so we have to change all entries that are strictly North-West of $(3,3)$:

$$M = \left(\begin{array}{cc|c} 3 & 2 & 1 \\ 3 & 3 & 0 \\ \hline 1 & 1 & \boxed{1} \end{array}\right) \longrightarrow M^{(3,3)} = \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 3 & 0 \\ \hline 1 & 1 & \boxed{1} \end{array}\right).$$

And then we continue

$$M^{(3,3)} = \left(\begin{array}{c|cc} 2 & 1 & 1 \\ 3 & 3 & 0 \\ \hline 1 & \boxed{1} & 1 \end{array}\right) \longrightarrow M^{(3,2)} = \left(\begin{array}{c|cc} 1 & 1 & 1 \\ 0 & 3 & 0 \\ \hline 1 & \boxed{1} & 1 \end{array}\right).$$

For the next step, observe that there is nothing strictly North-West of the box $(3,1)$. Hence

$$M^{(3,2)} = \left(\begin{array}{|ccc} 1 & 1 & 1 \\ 0 & 3 & 0 \\ \hline \boxed{1} & 1 & 1 \end{array}\right) \longrightarrow M^{(3,1)} = \left(\begin{array}{|ccc} 1 & 1 & 1 \\ 0 & 3 & 0 \\ \hline \boxed{1} & 1 & 1 \end{array}\right).$$

For the next step, the pivot is in position $(2,3)$. As the pivot is 0, nothing is changing, ie:

$$M^{(3,1)} = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ \hline 0 & 3 & \boxed{0} \\ 1 & 1 & 1 \end{array}\right) \longrightarrow M^{(2,3)} = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ \hline 0 & 3 & \boxed{0} \\ 1 & 1 & 1 \end{array}\right).$$

For the next step, the pivot is in position $(2,2)$. As the pivot is nonzero, we have to change the entries that are stictly North-West of $(2,2)$:

$$M^{(2,3)} = \left(\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 0 & \boxed{3} & 0 \\ 1 & 1 & 1 \end{array}\right) \longrightarrow M^{(2,2)} = \left(\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 0 & \boxed{3} & 0 \\ 1 & 1 & 1 \end{array}\right).$$

The last few steps are trivial as in each case there is nothing strictly North-West of the pivot. Hence we have:

$$M^{(1,1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

TNN Matrices and DD algorithm

Recall that $X = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(\mathbb{R})$ is a Cauchon matrix associated to the Cauchon diagram C provided that for all $(i, \alpha) \in [1, m] \times [1, p]$, we have $x_{i,\alpha} = 0$ if and only if $(i, \alpha) \in C$.

Goodearl-L.-Lenagan Let M be a matrix with real entries. We can apply the deleting derivation algorithm to M. Let $N = M^{(1,1)}$ denote the resulting matrix.
Then M is TNN iff the matrix N is nonnegative and Cauchon.

An example

Set $M = \begin{pmatrix} 11 & 4 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. Then $M^{(3,3)} = \begin{pmatrix} 7 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$,

$M^{(3,1)} = M^{(3,2)} = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, $M^{(2,3)} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$, and

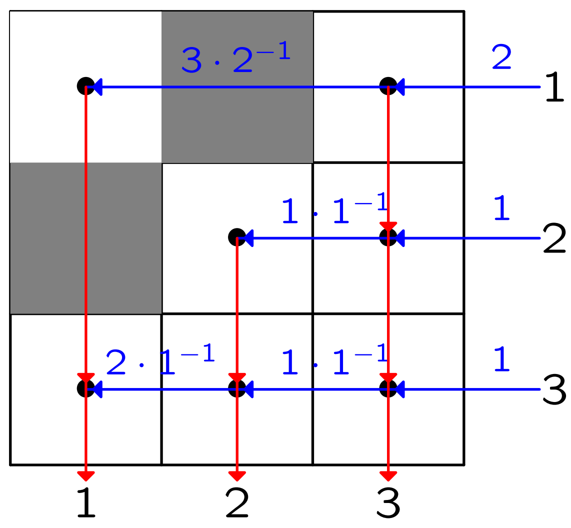
$$M^{(1,1)} = M^{(1,2)} = M^{(1,3)} = M^{(2,1)} = M^{(2,2)} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

So M is TNN as $M^{(1,1)}$ is nonnegative and its zeroes form a Cauchon diagram.

Application 1: new proof of Brenti's Theorem

Recall that $M = \begin{pmatrix} 11 & 4 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ is TNN and $M^{(1,1)} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

From $M^{(1,1)}$ we can deduce the following weighted planar network



The underlying unweighted planar network comes from Postnikov's work

The weights come from $M^{(1,1)}$

M is its weighted path matrix

TNN cells

Goodearl-L.-Lenagan Let M and N be two real $m \times p$ matrices. Then M and N are TNN and in the same cell if and only if $M^{(1,1)}$ and $N^{(1,1)}$ are nonnegative and Cauchon associated to the same Cauchon diagram.

So the TNN cells are the fibres of the map π that sends a TNN matrix M to the Cauchon diagram associated to $M^{(1,1)}$.

$\pi^{-1}(C)$ is the *TNN cell associated to the Cauchon diagram* C.

The TP cell corresponds to the all white Cauchon diagram, ie a matrix M is TP iff $M^{(1,1)}$ is positive.

Approximation of TNN matrices by TP matrices

$$\begin{array}{ccc} M \ \textbf{TNN} & \xrightarrow{\textbf{DD}} & M^{(1,1)} \ \textbf{nonnegative + Cauchon} \\ & & \Big\downarrow \ \text{0 replaced by } \varepsilon > 0 \\ N_\varepsilon \ \textbf{TP} & \xleftarrow[\textbf{R}]{} & N_\varepsilon^{(1,1)} \ \textbf{positive} \end{array}$$

Problem: N_ε does NOT tend to M when ε tends to 0.

Example: $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = M^{(1,1)}$, $\quad N_\epsilon^{(1,1)} = \begin{pmatrix} 1 & \epsilon \\ 1 & \epsilon \end{pmatrix}$, from which the restoration algorithm produces $N_\epsilon = \begin{pmatrix} 2 & \epsilon \\ 1 & \epsilon \end{pmatrix}$.

Approximation of TNN matrices by TP matrices

$$
\begin{array}{ccc}
M \ \textbf{TNN} & \xrightarrow{\ \textbf{DD}\ } & M^{(1,1)} \ \textbf{nonnegative} + \textbf{Cauchon} \\
\Big\uparrow \ \varepsilon \to 0 & & \Big\downarrow \ \text{0 in position } (i,j) \text{ replaced by } \varepsilon^{2^{(m-i)p+(p-j)}} \\
N_\varepsilon \ \textbf{TP} & \xleftarrow[\ \textbf{R}\ ]{} & N_\varepsilon^{(1,1)} \ \textbf{positive}
\end{array}
$$

Approximation of TNN matrices by TP matrices

$$M := \begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{(1,1)} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad N_\epsilon^{(1,1)} = \begin{pmatrix} 1 & \epsilon^{1024} & 1 \\ 1 & 1 & 1 \\ \epsilon^{32} & \epsilon^{16} & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

from which the restoration algorithm produces

$$N_\epsilon = \begin{pmatrix} 4 + 2\epsilon^{16} + \epsilon^{32} + 2\epsilon^{1024} + \epsilon^{1040} & 2 + \epsilon^{1024} + \epsilon^{16} & 1 \\ 3 + 2\epsilon^{16} + \epsilon^{32} & 2 + \epsilon^{16} & 1 \\ 1 + \epsilon^{16} + \epsilon^{32} & 1 + \epsilon^{16} & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

TNN versus Quantum

Goodearl-L.-Lenagan (2011) Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:

1. The totally nonnegative cell associated to $\mathcal{F}$ is non-empty.

2. $\mathcal{F}_q$ is the set of all quantum minors that belong to torus-invariant prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

Application: TNN test

Theorem (Gasca and Peña, 1992) A matrix is totally positive if each of its **initial minors** is positive.

Initial minors: solid minors with $i = 0$ or $j = 0$.

Examples: $[1, 2, 3 \mid 2, 3, 4]$ is initial, whereas $[2, 3, 4 \mid 2, 3, 4]$ isn't.

In the following, we give a criterion for a real matrix to be TNN and belong to a given cell. Our criterion generalises Gasca and Peña's Theorem.

Lacunary sequence

Let C be a Cauchon diagram. We say that a sequence

$$((i_0, \alpha_0),\ (i_1, \alpha_1), ...,\ (i_t, \alpha_t))$$

is a lacunary sequence with respect to C if the following conditions hold:

1. $t \geq 0$;

2. the boxes (i_1, α_1), (i_2, α_2), ..., (i_t, α_t) are white in C;

3. $1 \leq i_0 < i_1 < \cdots < i_t \leq m$ and $1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_t \leq p$;

4. If $i_t < i \leq m$ and $\alpha_t < \alpha \leq p$, then (i, α) is a black box in C;

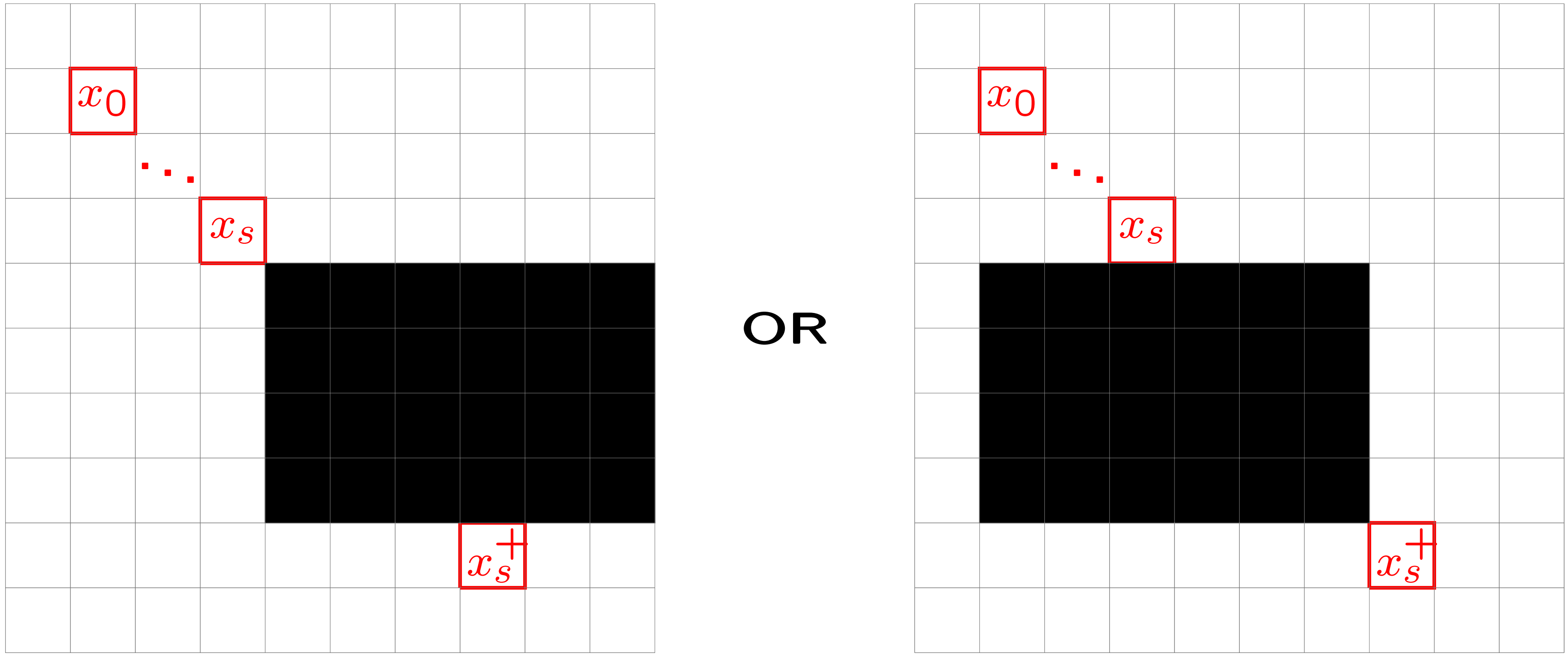
Lacunary sequence: Axiom 5

5. Let $s \in \{0, \ldots, t-1\}$. Then:

   - either (i, α) is a black box in C for all $i_s < i < i_{s+1}$ and $\alpha_s < \alpha$,

   - or (i, α) is a black box in C for all $i_s < i < i_{s+1}$ and $\alpha_0 \leq \alpha < \alpha_{s+1}$;

Axiom 5



where $x_s^+ := x_{s+1}$ and $x_k := (i_k, \alpha_k)$.

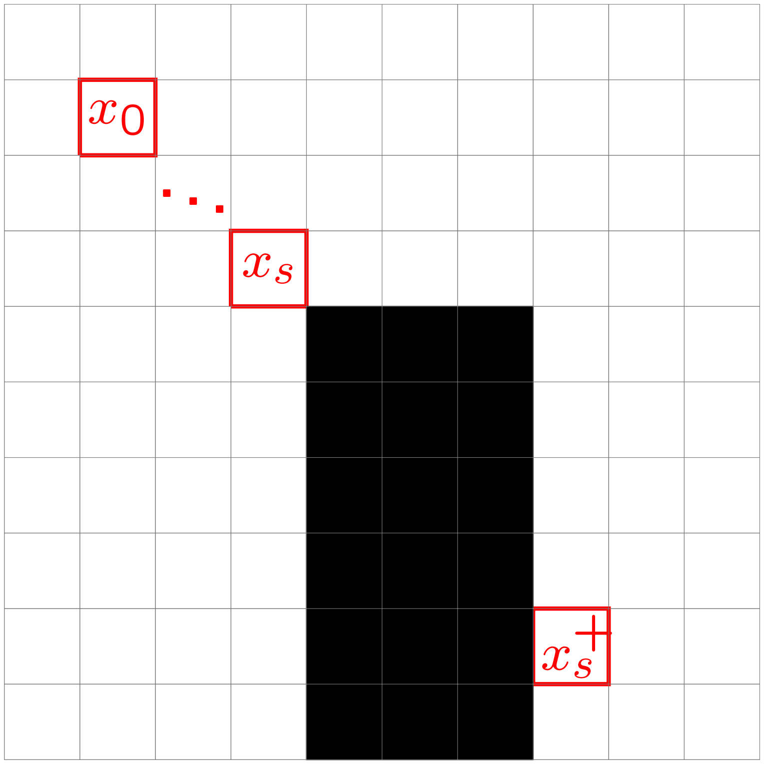
Lacunary sequence: Axiom 6

6. Let $s \in \{0, \ldots, t-1\}$. Then:

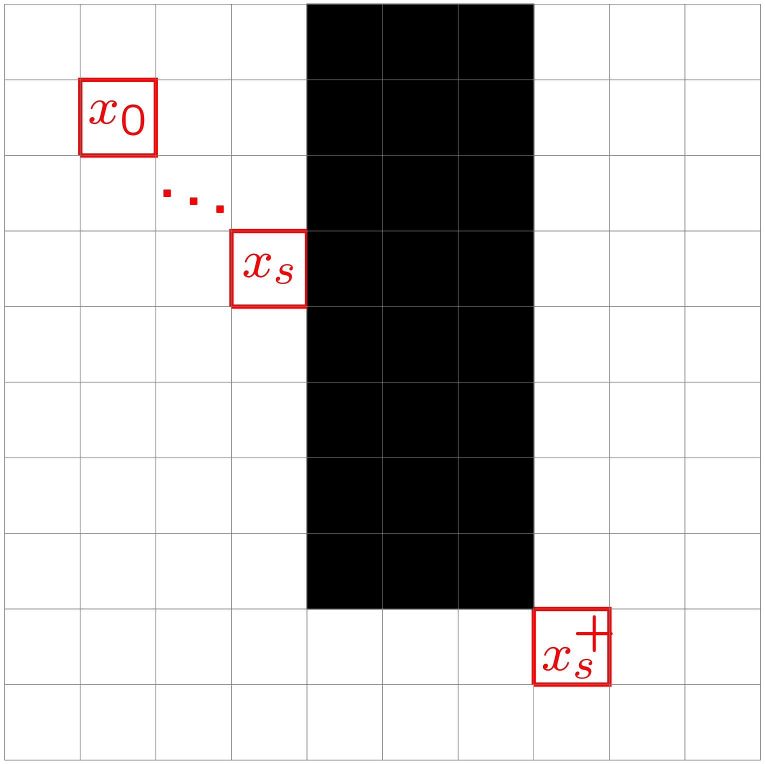
   - either (i, α) is a black box in C for all $i_s < i$ and $\alpha_s < \alpha < \alpha_{s+1}$,

   - or (i, α) is a black box in C for all $i < i_{s+1}$ and $\alpha_s < \alpha < \alpha_{s+1}$.

Axiom 6

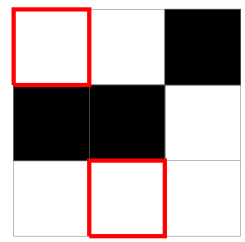


OR



where $x_s^+ := x_{s+1}$.

Example of lacunary sequence

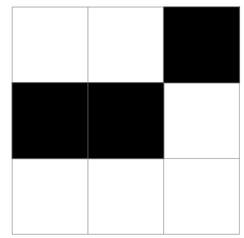
$C =$ 

One easily checks that $((1, 1),\ (3, 2))$ is a lacunary sequence starting at $(1, 1)$. Note however that $((1, 1),\ (2, 3))$ and $((1, 1),\ (3, 3))$ are not lacunary sequences.

Existence of lacunary sequences

Lemma Fix a Cauchon diagram C. Then for any $(j, \beta) \in [1, m] \times [1, p]$, there exists a lacunary sequence $((j, \beta),\ (i_1, \alpha_1), \ldots,\ (i_t, \alpha_t))$ starting at (j, β).

An example

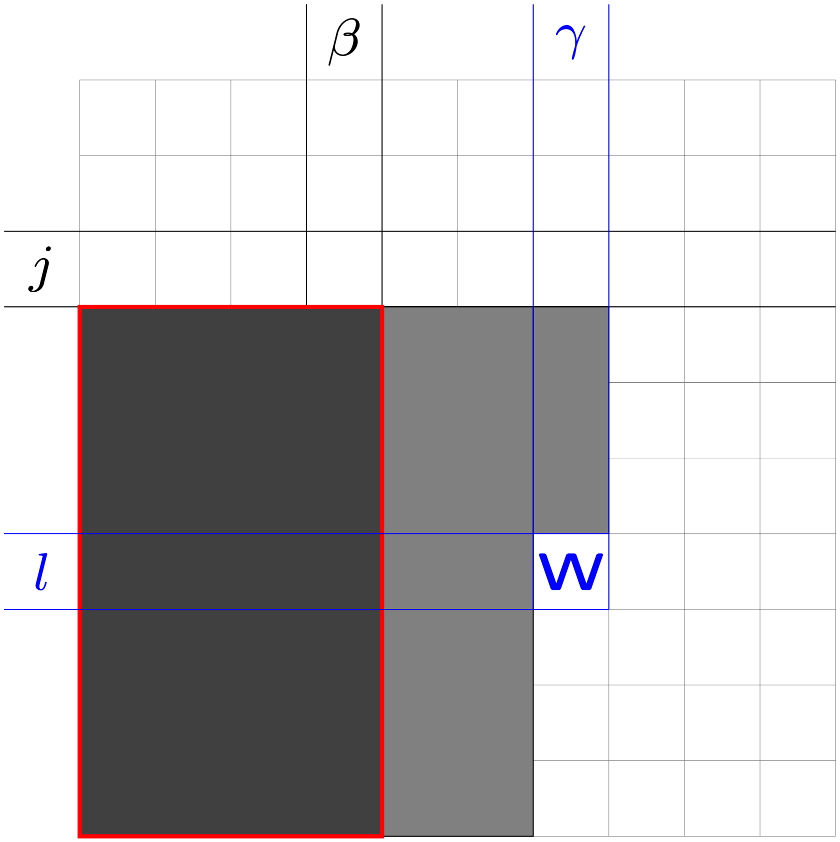
$C =$ 

$((1,1),(3,2))$	$((1,2),(2,3))$ or $((1,2),(3,3))$	$((1,3))$
$((2,1),(3,2))$	$((2,2),(3,3))$	$((2,3))$
$((3,1))$	$((3,2))$	$((3,3))$

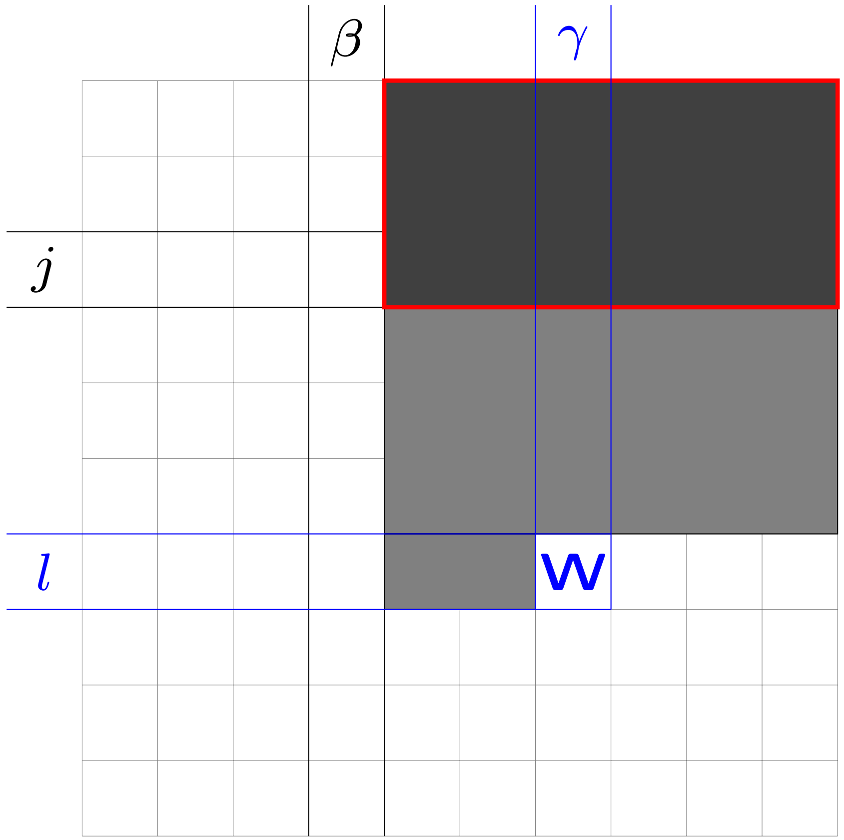
Existence of lacunary sequences

There is actually an algorithm that produces a lacunary sequence starting at any box. For, imagine that we have started constructing a lacunary sequence: $((i_0, \alpha_0),\ (i_1, \alpha_1), ...,\ (i_t, \alpha_t) = (j, \beta))$. And assume that there is a white box which is strictly south-east of (j, β) (so that the above is not a lacunary sequence). Then we can construct the next element in the sequence by distinguishing between 3 cases.

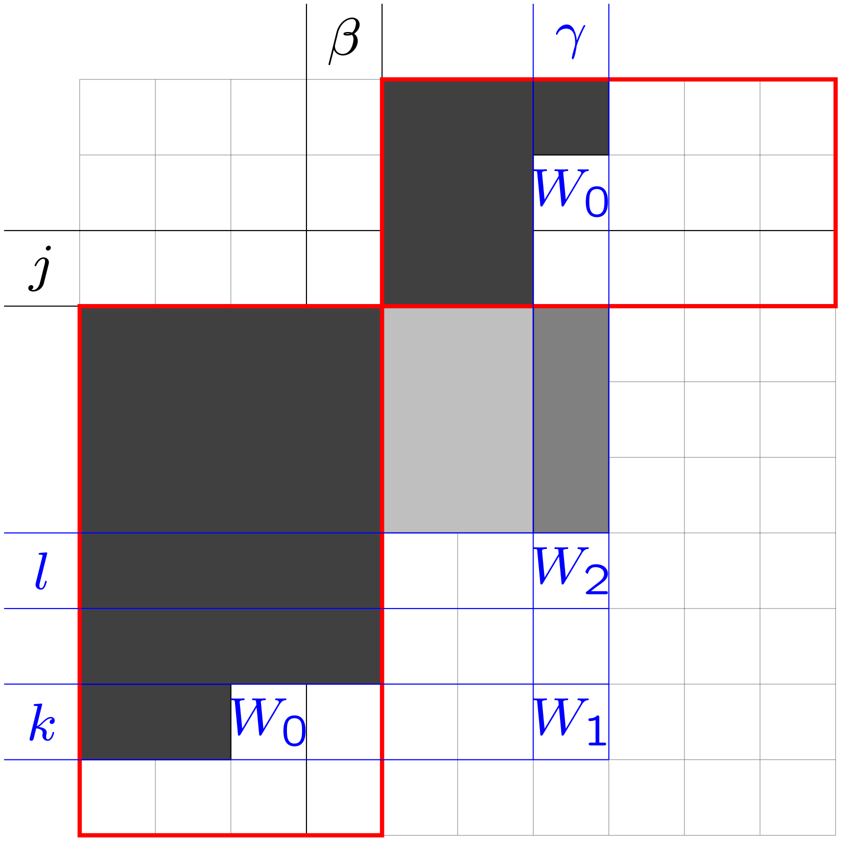
Case 1: all boxes (i, α) with $i > j$ and $\alpha \leq \beta$ are black



Case 2: all boxes (i, α) with $i \leq j$ and $\alpha > \beta$ are black



Case 3: we are not in cases 1 nor 2



TNN criteria

Fix a Cauchon diagram C. For all $(j,\beta) \in [1,m] \times [1,p]$, choose a lacunary sequence $((j,\beta),\ (i_1,\alpha_1),\ldots,\ (i_t,\alpha_t))$ starting at (j,β), and set

$$\Delta^C_{j,\beta} := [j < i_1 < \cdots < i_t \mid \beta < \alpha_1 < \cdots < \alpha_t] \in \mathcal{O}\left(\mathcal{M}_{m,p}(\mathbb{C})\right).$$

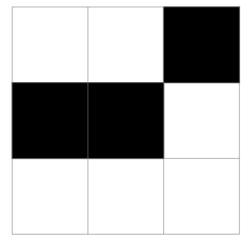
L.-Lenagan Let $M \in \mathcal{M}_{m,p}(\mathbb{R})$. TFAE

1. M is TNN and belongs to the TNN cell parametrised by C.

2. For all $(j,\beta) \in [1,m]\times[1,p]$, we have $\Delta^C_{j,\beta}(M) = 0$ if $(j,\beta) \in C$ and $\Delta^C_{j,\beta}(M) > 0$ if $(j,\beta) \notin C$.

This test only involves $m \times p$ minors. This generalises the result of Gasca and Peña.

An example

$C =$ 

$\Delta_{1,1} = [13|12] > 0$ $\Delta_{1,2} = [12|23]$ or $[13|23] > 0$ $\Delta_{1,3} = [1|3] = 0$

$\Delta_{2,1} = [23|12] = 0$ $\Delta_{2,2} = [23|23] = 0$ $\Delta_{2,3} = [2|3] > 0$

$\Delta_{3,1} = [3|1] > 0$ $\Delta_{3,2} = [3|2] > 0$ $\Delta_{3,3} = [3|3] > 0$

TNN criteria: sketch of proof

Fix a Cauchon diagram C. For all $(j,\beta) \in [1,m] \times [1,p]$, choose a lacunary sequence $((j,\beta),\ (i_1,\alpha_1),\ldots,\ (i_t,\alpha_t))$ starting at (j,β), and set

$$\Delta^C_{j,\beta} := [j < i_1 < \cdots < i_t \mid \beta < \alpha_1 < \cdots < \alpha_t] \in \mathcal{O}\left(\mathcal{M}_{m,p}(\mathbb{C})\right).$$

Let $M \in \mathcal{M}_{m,p}(\mathbb{R})$. If one of the following conditions is satisfied

1. M is TNN and belongs to the TNN cell parametrised by C;

2. For all $(j,\beta) \in [1,m] \times [1,p]$, we have $\Delta^C_{j,\beta}(M) = 0$ if $(j,\beta) \in C$ and $\Delta^C_{j,\beta}(M) > 0$ if $(j,\beta) \notin C$;

then

$$\Delta^C_{j,\beta}(M) = t_{j,\beta} \cdot t_{i_1,\alpha_1} \cdots t_{i_t,\alpha_t},$$

where $M^{(1,1)} = (t_{i,\alpha})$.